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Spontaneous Translation Symmetry Breakdown for Quantum Lattice Systems

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We study the set of equilibrium states for quantum lattice states in the presence of a translation symmetry of the model. We derive a characterization of the spontaneous breaking of this symmetry, i.e., the decomposition of an invariant equilibrium state into a mixture of noninvariant equilibrium states, in terms of the separability in mean energy of these states for a class of perturbed dynamics.

KEY WORDS: Equilibrium states; Choquet simplex; invariant mean; symmetry breakdown.

1. INTRODUCTION

There are several alternative methods of describing the states that occur in statistical mechanics. One of these methods is to identify the states as normalized positive linear functionals over a C^* algebra of kinematic observables $\mathfrak{A}$. If we consider, in this context, a system on a lattice Z^ν, the group of space translations Z^ν acts as a group of automorphisms $\{\tau_x, x \in Z^\nu\}$ on the algebra $\mathfrak{A}$ and therefore on the set of states.

It has been established that the equilibrium states at a given temperature form a Choquet simplex, i.e., a convex set such that each element has a unique barycentric decomposition in terms of the extremal elements of the set.

This decomposition appears to correspond to the physical separation of an equilibrium state into pure thermodynamic phases.

The set of equilibrium states corresponding to a translationally invariant interaction is clearly globally invariant by space translations, and

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therefore by a well-know fixed-point theorem the subset of invariant equilibrium states is nonempty.

We are concerned with the problem of the decomposition of the (translation-) invariant equilibrium states into convex combinations of extremal equilibrium states, namely, under what conditions some translation-invariant equilibrium state ρ is a mixture of translation-noninvariant extremal equilibrium states (pure thermodynamical phases).

It is the general belief that this situation occurs if, under the influence of small perturbations, the system can leave the state ρ and arrive in an equilibrium situation which is not invariant under translations of the system. Then one says that the translational symmetry is broken.

Our main result supports these ideas by showing that the translational symmetry is broken if and only if there exists a perturbation of the dynamics of a certain type which can change the energy density of the system (Theorem 4.2 below). The opposite situation is therefore the case in which translation invariance is very stable.

If we take the example of the Ising ferromagnetic model, e.g., Ref. 1, for which it is known that in three dimmensions there are translation-noninvariant equilibrium states,[2] it is known that the symmetry is not broken.

From the geometrical point of view, broken symmetry is an extreme case, which, for instance, cannot appear in a finite-dimensional Choquet simplex.

For classical lattice systems in the case in which the set of translation-invariant equilibrium states reduces to one point, the same problem was considered in Ref. 3, but using a different approach.

In a previous paper,[4] the authors considered a locally compact amenable group G, acting as affine homeomorphisms of a Choquet simplex Δ. Theorem 4.2 of Ref. 4 characterizes the extreme case where the set of fixed points of Δ by the action of G is a face of Δ, by the following stability condition: for all points $\omega \in \Delta$, all invariant means M on G and all continuous functions f on Δ we have

$$M(|f(\omega) - f(\tau_g \omega)|) = 0$$

where τ_g denotes the action of $g \in G$ on Δ.

Here we give an application of this result in the case where the simplex Δ is the set of equilibrium states of a quantum lattice system on Z^ν.

Recall (Ref. 6), that for such a model one associates to each finite region X of Z^ν the algebra of all bounded linear operators on a finite-dimensional Hilbert space $\mathscr{H}(X)$ and the algebra of all observables is then defined by

$$\mathfrak{A} = \overline{\bigcup_{X \subset Z^\nu} \mathfrak{A}(X)}$$

The group of lattice translations Z^ν acts as a group of automorphisms of $\mathfrak{A}$ and we denote this action by

$$A \in \mathfrak{A}(X) \to \tau_x A \in \mathfrak{A}(X+x), \qquad x \in Z^\nu$$

In this context an invariant interaction is a function Φ from the set of finite subsets of Z^ν to the Hermitian elements of $\mathfrak{A}$, such that

$$\Phi(X) \in \mathfrak{A}(X) \qquad \text{and} \qquad \Phi(X+x) = \tau_x \Phi(X)$$

We consider interactions for which the dynamics is well defined and in that case the set of equilibrium states at a given temperature, defined for instance via the Kubo, Martin, Schwinger condition, is a Choquet simplex.

In what follows a mean on Z^ν should be interpreted as a normalized linear functional on the C^* algebra of all bounded complex-valued functions on Z^ν, equipped with the sup norm $\|f\|_\infty$.

A mean M on Z^ν is invariant if $M(f) = M({}_x f)$, $x \in Z^\nu$ where

$${}_x f(a) = f(a-x), \qquad a \in Z^\nu$$

and it is extremal invariant if it is an extreme point of the convex set of invariant means.

2. UNIFORMLY AVERAGEABLE INTERACTIONS

In this section we introduce interactions which are not invariant for translations.

An interaction Φ of the infinite lattice system is called uniformly finite range if there exists a finite subset Λ of Z^ν such that for any finite $X \subset Z^\nu$, $\Phi(X) = 0$ implies $X \subset \Lambda + x$ for some $x \in Z^\nu$.

Denote by $\mathscr{A}_0$ the space of all interactions Φ which are uniformly finite range and for which

$$\|\Phi\| = \sup_{x \in Z^\nu} \sum_{X \ni x} \frac{\|\Phi(X)\|}{N(X)} \tag{2.1}$$

is finite.

(2.1) is a norm on $\mathscr{A}_0$. The completion $\mathscr{A}$ of $\mathscr{A}_0$ for this norm is a Banach space of interactions. $\mathscr{B}$ is defined as the subspace of $\mathscr{A}$ containing the interactions which are invariant for translations.

For any $\Phi \in \mathscr{A}$, for any $x \in Z^\nu$ the energy at the point x is defined by

$$A_\Phi^x = \sum_{X \ni x} \frac{\Phi(x)}{N(x)} \tag{2.2}$$

In this way the energy per point is an observable in the C^* algebra.

In the following we will have to take limits on increasing subsets of Z^ν. Consider a sequence $(\Lambda_n)_{n \in \mathbb{Z}}$ of finite subsets of Z^ν satisfying, for any $x \in Z^\nu$,

$$\lim_{n \to \infty} \frac{N(\Lambda_n \Delta(\Lambda_n + x))}{N(\Lambda_n)} \text{ converges to zero} \tag{2.3}$$

[Δ is the symmetric difference, $N(\Lambda)$ is the number of points in Λ.]

Any time we write down $\lim_{\Lambda \to \infty}$ we mean $\lim_{n \to \infty}$ for any sequence $(\Lambda_n)_n$ satisfying (2.3) and the limit point is independent of this sequence.

Remark. This is equivalent to the limit in the sense of van Hove.

We introduce the following:

Definition 2.1. A bounded real function f on Z^ν is uniformly averageable if there exists a constant $m(f)$ such that the limit

$$m(f) = \lim_{\Lambda \to \infty} \frac{1}{N(\Lambda)} \sum_{x \in \Lambda} {}_x f$$

converges uniformly on Z^ν. [${}_x f$ is the function on Z^ν defined by ${}_x f(y) = f(y-x)$ for any $y \in Z^\nu$.]

In particular $m(f)$ belongs to the closed convex hull of the set $\{{}_x f, x \in Z^\nu\}$. Hence for any invariant mean M on Z^ν one has $M(f) = m(f)$.

Remark. Clearly $g(x) = f(x) - m(f)$ has asymptotically vanishing average in the following sense:

$$\lim_{\Lambda \to \infty} \sup_y \frac{1}{N(\Lambda)} \left| \sum_{x \in \Lambda} g(y-x) \right| = 0$$

Definition 2.2. An interaction $\Phi \in \mathscr{A}$ is *uniformly averageable* if for any finite subset X of Z^ν there exists a uniformly averageable function f^X on Z^ν such that

$$\Phi(X + x) = f^X(x)\, \tau_x \Phi(X) \qquad \text{for any } x \in Z^\nu$$

Denote by $\mathscr{A}^A$ the class of uniformly averageable interactions. Notice that $\mathscr{B} \subset \mathscr{A}^A$. Furthermore $\mathscr{A}^A$ is a Banach subspace of $\mathscr{A}$ because the set of uniformly averageable functions on Z^ν is uniformly closed.

Theorem 2.3. Let $\Phi \in \mathscr{A}^A$. Let ω be a translation invariant state on $\mathfrak{A}$. Then the following holds: (a) The function $x \in Z^v \mapsto \omega(A_\Phi^x)$ is uniformly averageable; (b) the limit

$$\varepsilon_\Phi(\omega) = \lim_{\Lambda \to \infty} \frac{1}{N(\Lambda)} \omega\left(\sum_{X \subset \Lambda} \Phi(X) \right)$$

exists and

$$\varepsilon_\Phi(\omega) = m(x \mapsto \omega(A_\Phi^x))$$

Proof. (a) Notice that

$$A_\Phi^x = \sum_{X \ni x} \frac{\Phi(X)}{N(X)} = \sum_{X \ni 0} \frac{\Phi(X+x)}{N(X)} = \sum_{X \ni 0} \frac{f^X(x)}{N(X)} \tau_x \Phi(x)$$

Hence

$$\omega(A_\Phi^x) = \sum_{X \ni 0} \frac{f^X(x)}{N(X)} \omega(\Phi(X)) \tag{2.4}$$

Suppose first that Φ is uniformly finite range. Then the function $x \in Z^v \mapsto \omega(A_\Phi^x)$ is a finite sum of uniformly averageable functions on Z^v, hence itself is uniformly averageable.

Because the uniformly finite range interactions are dense in $\mathscr{A}^A$ and because the space of uniformly averageable functions on Z^v is closed (a) follows for arbitrary $\Phi \in \mathscr{A}^A$.

(b) Again it is enough to prove (b) in the case in which Φ is uniformly finite range. Let $\varepsilon > 0$. Let $\Lambda \subset Z^v$. Put

$$H_\Lambda(\Phi) = \sum_{X \subset \Lambda} \Phi(x)$$

Suppose

$$\frac{N(\Lambda \, \Delta(\Lambda + x))}{N(\Lambda)} < \varepsilon \qquad \text{for any } x \in \Lambda_0$$

Then

$$\frac{1}{N(\Lambda)} \left\| H_\Lambda(\Phi) - \sum_{x \in \Lambda} A_\Phi^x \right\| < N(\Lambda_0)\varepsilon \|\Phi\| \tag{2.5}$$

This shows

$$\lim \frac{1}{N(\Lambda)} \left\| H_\Lambda(\Phi) - \sum_{x \in \Lambda} A_\Phi^x \right\| = 0 \tag{2.6}$$

Hence (b) follows from (a). ∎

3. STATES SEPARATED IN MEAN

Two translation-invariant states on $\mathfrak{A}$ can be separated by the energy density functional ε_Φ for some interaction Φ in $\mathscr{B}$. The aim of this section is to prove results of this kind for states which are not invariant under translations using the classes $\mathscr{A}^A$ and $\mathscr{A}$ of interactions.

The main result of this section is the following:

Theorem 3.1. Let ω_1 and ω_2 be states on $\mathfrak{A}$. Let M be an extremally invariant mean on Z^ν. The following are equivalent: (a) for any $A \in \mathfrak{A}$

$$M(x \in Z^\nu \mapsto |\omega_1(\tau_x A) - \omega_2(\tau_x A)|^2) = 0 \tag{3.1}$$

(b) [resp. (c)] for any $\Phi \in \mathscr{A}^A$ (resp. $\Phi \in \mathscr{A}$) the following energy densities are equal:

$$M(x \in Z^\nu \mapsto \omega_1(A_\Phi^x)) = M(x \in Z^\nu \mapsto \omega_2(A_\Phi^x)) \tag{3.2}$$

Proof:

(a) $\Rightarrow$ (c)

Let $\Phi \in \mathscr{A}$ and let Λ be a finite subset of Z^ν. Let $(u_{ij})_{ij}$ be matrix units for the C^* algebra $\mathfrak{A}(\Lambda)$. Define functions f_{ij} on Z^ν by

$$\Phi(\Lambda + x) = \sum f_{ij}(x)\, \tau_x u_{ij} \qquad \text{for any } x \in Z^\nu$$

The functions f_{ij} are bounded by $\|\Phi\|$. Hence

$$\begin{aligned} |M(x \mapsto (\omega_1 - \omega_2)(\Phi(\Lambda + x))| &\leq \sum_{ij} |M(x \mapsto f_{ij}(x)(\omega_1 - \omega_2)(\tau_x u_{ij})| \\ &\leq \|\Phi\| \sum_{ij} M(x \mapsto |(\omega_1 - \omega_2)(\tau_x u_{ij})|^2)^{1/2} \end{aligned}$$

which equals zero by condition a). Hence:

$$M(x \mapsto \omega_1(\Phi(\Lambda + x)) = M(x \mapsto \omega_2(\Phi(\Lambda + x)) \tag{3.3}$$

for any finite subset Λ of Z^ν.

Suppose first Φ is uniformly finite range. There exists a finite subset Λ_0 of Z^ν such that

$$A_\Phi^x = \sum_{0 \in X \subset \Lambda_0} \Phi(X + x) \qquad \text{for any } x \in Z^\nu$$

Then (3.3) implies (3.2). Because the uniformly finite range interactions are dense in $\mathscr{A}$, (3.2) follows for abitrary $\Phi \in \mathscr{A}$.

$(c) \Rightarrow$ (b) is trivial.

Essential in the proof of (b) $\Rightarrow$ (a) is the following:

Lemma 3.2. Let A be a self-adjoint element of some local algebra $\mathfrak{A}(\Lambda)$. Let f be an uniformly averageable function on Z^v. Let $y \in Z^v \backslash \Lambda$. There exists a $\Phi \in \mathscr{A}^A$ such that

$$A_{\Phi}^{x+y} = f(x)\, \tau_x A \tag{3.4}$$

for any $x \in Z^v$.

Proof. Define an interaction Φ by

$$\begin{aligned} \Phi(\Lambda + x) &= -N(\Lambda) f(x)\, \tau_x A, && x \in Z^v \\ \Phi(\Lambda \cup \{y\} + x) &= [N(\Lambda) + 1] f(x)\, \tau_x A, && x \in Z^v \\ \Phi(X) &= 0, && \text{otherwise} \end{aligned} \tag{3.5}$$

Then $\Phi \in \mathscr{A}^A$ and Φ satisfies (3.4). ∎

Proof of (b) $\Rightarrow$ *(a) in Theorem 3.1.* It is enough to prove (3.1) for any self-adjoint element A of any local algebra $\mathfrak{A}(\Lambda)$.

Let f be a uniformly averageable function on Z^v. Take $y \in Z^v \backslash \Lambda$.

Let Φ be the interaction constructed in Lemma 3.2. Then condition (b) of the theorem yields

$$M(x \in Z^v \mapsto f(x)(\omega_1 - \omega_2)(A_{\Phi}^x)) = 0 \tag{3.6}$$

Denote $g(x) = (\omega_1 - \omega_2)(A_{\Phi}^x)$ for any $x \in Z^v$. Then g is a bounded function on Z^v. Let $z \in Z^v$. Take $f = g - {}_z g$. Then f is uniformly averageable.

By (3.6) one obtains $M((g - {}_z g)\, g) = 0$. Or

$$M(g \,.\, {}_z g) = M(g^2) \qquad \text{for any } z \in Z^v \tag{3.7}$$

It has been proved in Ref. 4 that this implies $M(g^2) = M(g)^2$.

But taking f constant in (3.6) it follows that $M(g) = 0$. Hence $M(g^2) = 0$. This exactly is equation (3.1). ∎

Corollary 3.3. Let ω and ρ be states on $\mathfrak{A}$. Suppose ρ is invariant under translations. Let M be an extremally invariant mean on Z^v. The following are equivalent: (a) For any $A \in \mathfrak{A}$

$$M(x \in Z^v \mapsto |\omega(\tau_x A) - \rho(A)|^2) = 0 \tag{3.8}$$

(b) For any $\Phi \in \mathscr{A}^{A}$

$$\varepsilon_{\Phi}(\rho) = M(x \mapsto \omega(A_{\Phi}^{x})) \tag{3.9}$$

Proof. This follows from the theorems 2.3 and 3.1. ∎

4. SYMMETRY BREAKDOWN

Now we came to the main point of our paper. By Theorem 4.2, below, any translation-invariant equilibrium state cannot be a mixture of translation noninvariant equilibrium states if and only if for any equilibrium state, there is a translation-invariant equilibrium state which cannot be separated in mean energy by any uniformly averageable potential.

In order to make the connection with Ref. 4, since Theorem 4.2 of Ref. 4 will be used in what follows, we recall the following definition:

Definition 4.1. A convex subset F of Δ is a *face* of Δ if for any $\rho \in F$, $\omega_1, \omega_2 \in \Delta$, $\lambda \in]0, 1[$ such that $\rho = \lambda\omega_1 + (1-\lambda)\omega_2$ one has $\omega_1, \omega_2 \in F$ (see Ref. 8).

From the above it is clear that the absence of spontaneous symmetry breakdown is equivalent to the fact that Δ_I, the convex subset of the Z^{ν}-invariant equilibrium states, is a face of Δ, the Choquet simplex of all equilibrium states.

Theorem 4.2. Let Δ, Δ_I be as above. The following equivalent:

(a) Δ_I is a face of Δ.

(b) For any state ω in Δ and for any extremally invariant mean M on Z^{υ} there exists a state $\rho \in \Delta_I$ for which

$$M(x \mapsto |\omega(\tau_x A) - \rho(A)|^2) = 0, \qquad A \in \mathfrak{A} \tag{4.1}$$

(c) For any state ω in Δ and for any invariant mean M on Z^{υ} there exists a state $\rho \in \Delta_I$ such that for any interaction $\Phi \in \mathscr{A}^{A}$, the following energy densities are equal:

$$\varepsilon_{\Phi}(\rho) = M(x \in Z^{\upsilon} \mapsto \omega(A_{\Phi}^{x})) \tag{4.2}$$

Proof. Let M be an invariant mean on Z^{υ}. The state ρ in conditions (b) and (c) is uniquely determined by $\rho(A) = M(x \mapsto \omega(\tau_x A))$ for any $A \in \mathfrak{A}$. Equation (4.1) now can be reformulated:

$$M(x \mapsto |\omega(\tau_x A)|^2) = |M(x \mapsto \omega(\tau_x A))|^2 \tag{4.3}$$

for any $A \in \mathfrak{A}$.

The equivalence of (a) and (b) now follows from Ref. 4, Theorem 4.2 (see the Introduction for the statement of this Theorem).

The equivalence of (b) and (c) follows from Corollary 3.3. Notice that Eq. (4.2) is affine in the mean M so that it holds for any invariant mean if it holds for any extremally invariant mean. ∎

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